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Higher-order Graph Cuts

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Labeling problem

Y X

Original | Noise-added | Denoised

- Assign a label to each pixel
  - Pixel $v \in V \rightarrow$ Label $X_v \in L$

Energy minimization

Y X

Original   Noise-added   Denoised

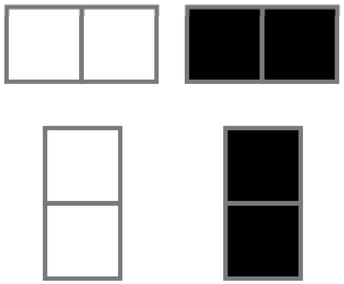
$$E(X) = \sum_{v} \lambda |Y_v - X_v| + \sum_{(u,v)} \kappa |X_u - X_v|$$

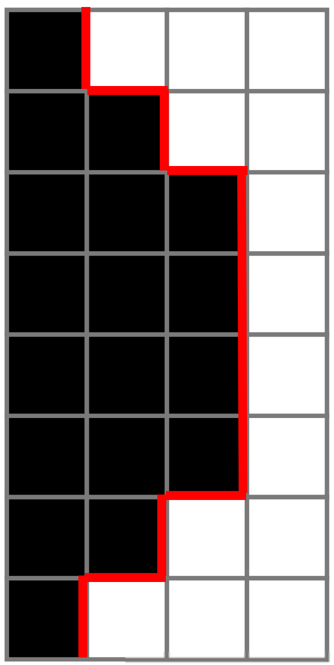
- Globally optimizeable using graph cuts

  Greig, Porteous and Seheult '89

First-order (pairwise) energy

Good (Low Energy)

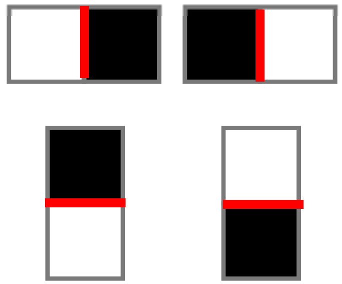


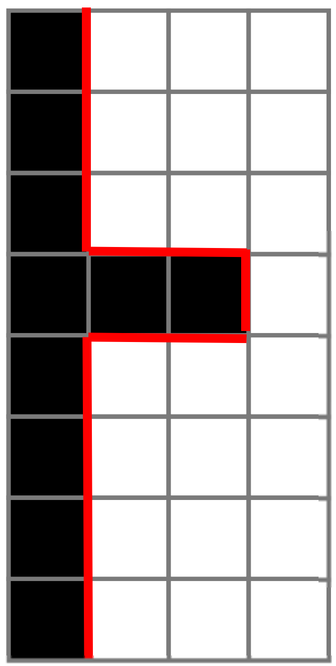


12 Bad

40 Good

Bad (High Energy)





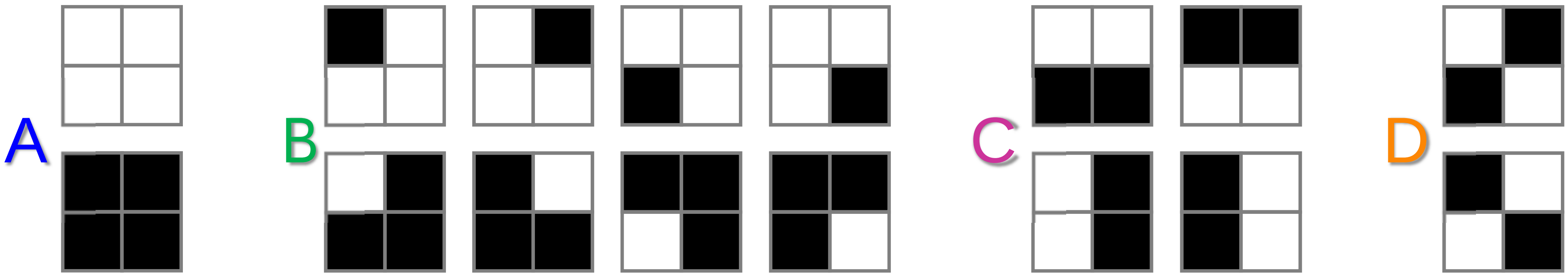
12 Bad

40 Good

Higher-order energy

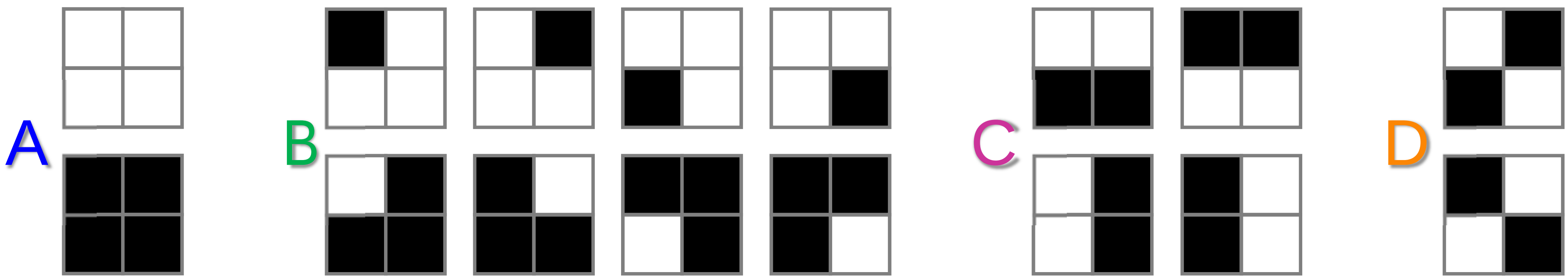
Good (Low Energy) Bad (High Energy)

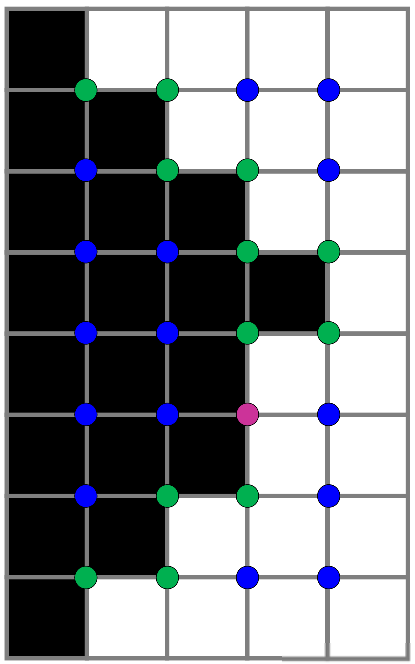
Better (Lower Energy) Worse (Higher Energy)



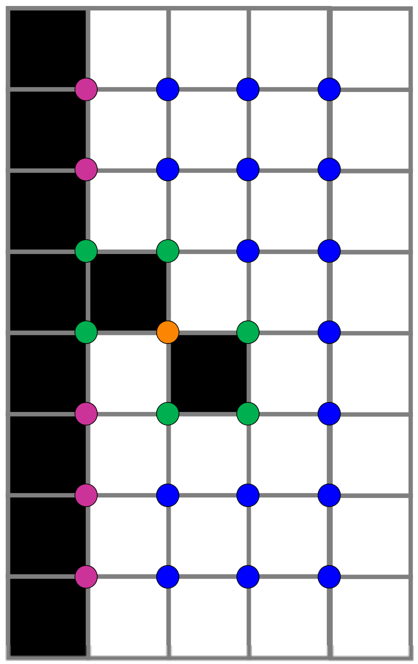
Higher-order energy

Better (Lower Energy) ⟷ Worse (Higher Energy)





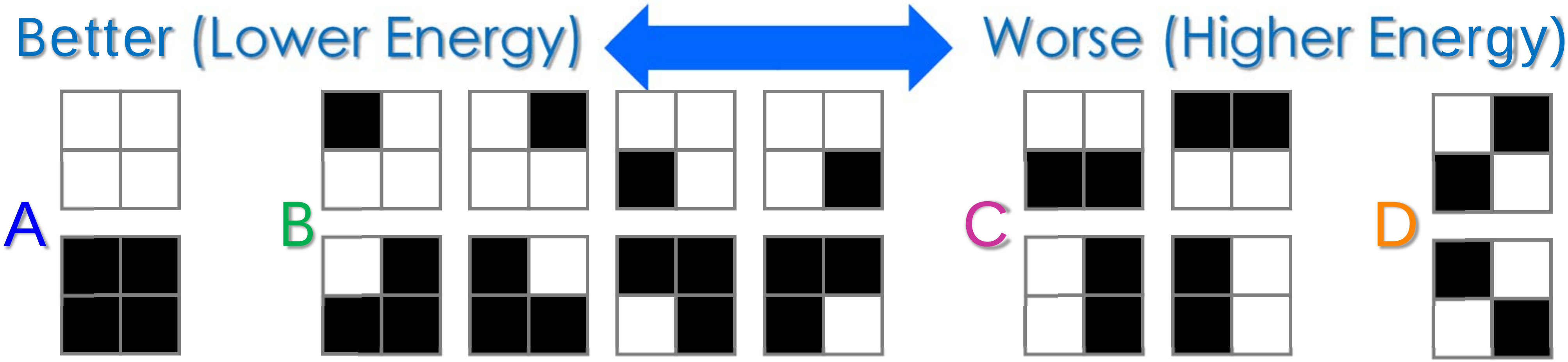
A:15
B: 12
C: 1
D: 0



A:16
B: 6
C: 5
D: 1

Higher-order energy

$$E(X) = \sum_{C \in \mathcal{C}} f_C(X_C)$$
$$= \sum_{v \in V} g_v(X_v) + \sum_{(u,v) \in E} h_{uv}(X_u, X_v)$$
$$+ \sum_{(u,v,s,t)} k_{uvst}\,(X_u, X_v, X_s, X_t)$$



Functions of binary variables

- Pseudo-Boolean function (PBF)
  - Function of binary (0 or 1) variables
  - Can always write it uniquely as a polynomial
- One variable x: $E_0(1-x)+E_1x$
- Two variables x, y:

$$E_{00}(1-x)(1-y)+E_{01}(1-x)y+E_{10}x(1-y)+E_{11}xy$$

- Three variables x, y, z:

$$E_{000}(1-x)(1-y)(1-z)+E_{001}(1-x)(1-y)z+\cdots+E_{111}xyz$$

- n^{th} order binary MRF = $(n+1)^{\text{th}}$ degree PBF

Reducing higher-order energy

- Convert any higher-order binary energy

$$E(X) = E(X_1, \dots, X_n) = \sum_{C \in \mathcal{C}} f_C(X_C)$$

into an equivalent first-order energy

$$\tilde{E}(X) = \tilde{E}(X_1, \dots, X_n, \dots, X_m) = \sum g_v(X_v) + \sum h_{uv}(X_u, X_v)$$

- Adds variables
- More than 2 labels → Fusion moves

Ishikawa CVPR 2009, PAMI 2011

New!

- Convert many higher-order binary energies

$$E(X) = E(X_1, \ldots, X_n) = \sum_{C \in \mathscr{C}} f_C(X_C)$$

into an equivalent first-order energy

$$\tilde{E}(X) = \tilde{E}(X_1, \ldots, X_n \qquad) = \sum g_v(X_v) + \sum h_{uv}(X_u, X_v)$$

- Without adding variables
- How? Ishikawa CVPR 2014

Example: a cubic term

- $\varphi(x, y, z)$: cubic (3rd degree)

$$\varphi(0,0,0) = a,\ \varphi(1,0,0) = b,\ \varphi(0,1,0) = c,\ \varphi(1,1,0) = d,$$
$$\varphi(0,0,1) = e,\ \varphi(1,0,1) = f,\ \varphi(0,1,1) = g,\ \varphi(1,1,1) = h$$

- $\varphi(x,y,z) = a(1-x)(1-y)(1-z) + bx(1-y)(1-z)$
$+ c(1-x)y(1-z) + dxy(1-z) + e(1-x)(1-y)z$
$+ fx(1-y)z + g(1-x)yz + hxyz$

- xyz coefficient: $s = -a + b + c - d + e - f - g + h$

- Define new function:

$$\varphi'(x, y, z) = \begin{cases} \varphi(x, y, z) & \text{when } (x, y, z) \neq (0,0,0) \\ \varphi(0,0,0) + s & \text{when } (x, y, z) = (0,0,0) \end{cases}$$

i.e., value is added s only when $(x, y, z) = (0,0,0)$

Example: a cubic term

- Define new function:

$$\varphi'(x, y, z) = \begin{cases} \varphi(x, y, z) & when\ (x, y, z) \neq (0,0,0) \\ \varphi(0,0,0) + s & when\ (x, y, z) = (0,0,0) \end{cases}$$

  i.e., value is added s only when $(x, y, z) = (0,0,0)$

- New xyz coefficient (replace a with $a + s$):

  $s' = -(a + s) + b + c - d + e - f - g + h = s - s = 0$

- So φ' is now quadratic (2nd degree)

- Similarly, we can reduce the degree by changing one of the 8 possible values

- But φ and φ' are different functions!

When can we do this?

$\varphi'(0,0,0) = a + s$, $\varphi'(1,0,0) = b$, $\varphi'(0,1,0) = c$, $\varphi'(1,1,0) = d$,
$\varphi'(0,0,1) = e$, $\varphi'(1,0,1) = f$, $\varphi'(0,1,1) = g$, $\varphi'(1,1,1) = h$

- Different only when $(x, y, z) = (0,0,0)$
- Suppose φ is a potential in $E(X) = \sum f_C(X_C)$
  - x, y, z are three of the variables in X
- If
  - $s > 0$, and
  - For minimizer X of $E(X)$, $(x, y, z) \neq (0,0,0)$,

then, we can replace φ with φ' without changing the minimizer

ELC

- Excludable Local Configuration (ELC)
  - A (usually) locally-testable sufficient condition for local configuration (x, y, z) to be not part of global minimizer
  - "Excludable as a part of global minimizer"
- ELC may not exist
- May take time to find
- Approximation
  - Just use the local configuration (x, y, z) with the largest value $\varphi(x, y, z)$

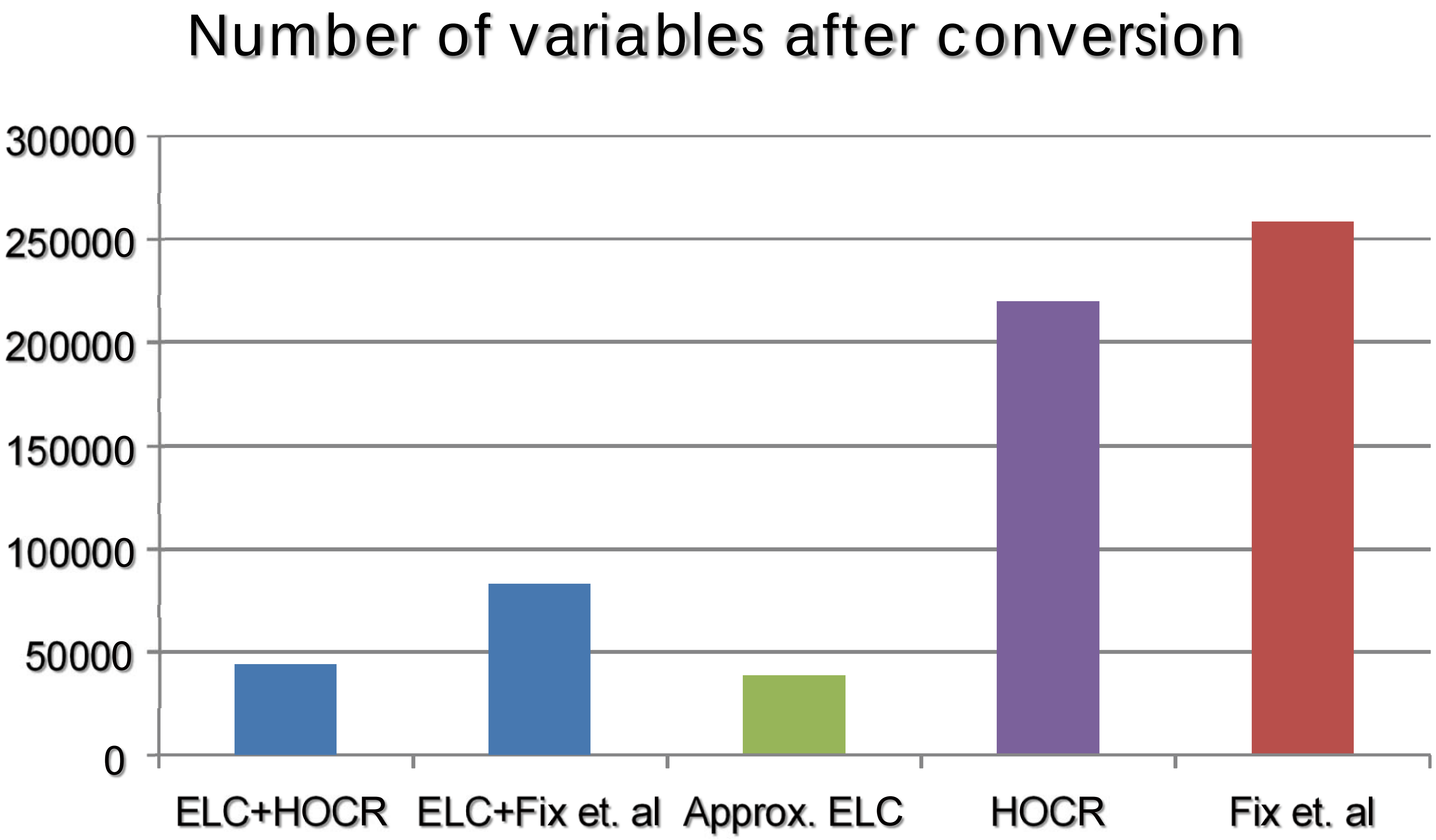
Experiment: 4th deg. FoE denoising

Original

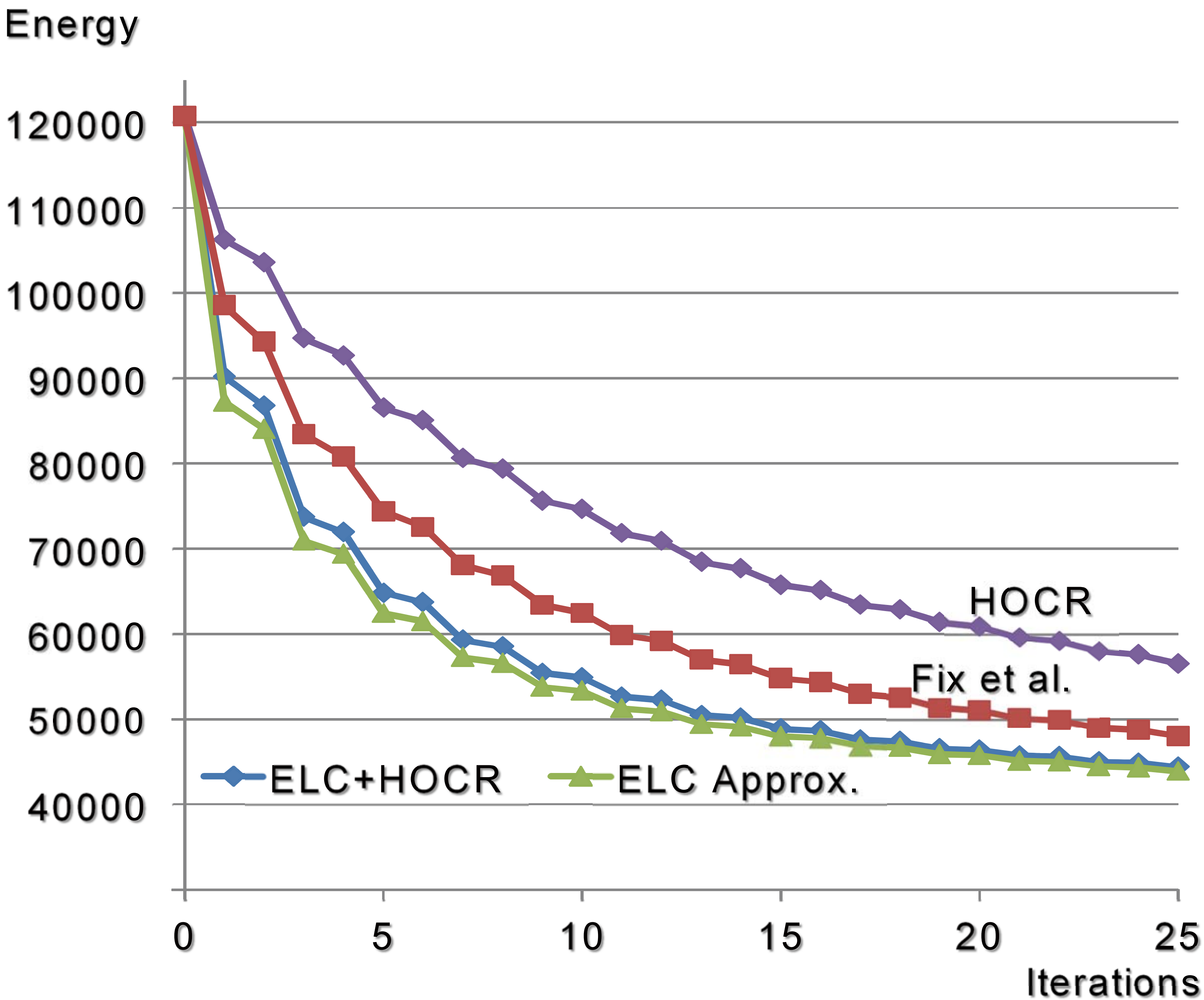
Noise added

4th degree

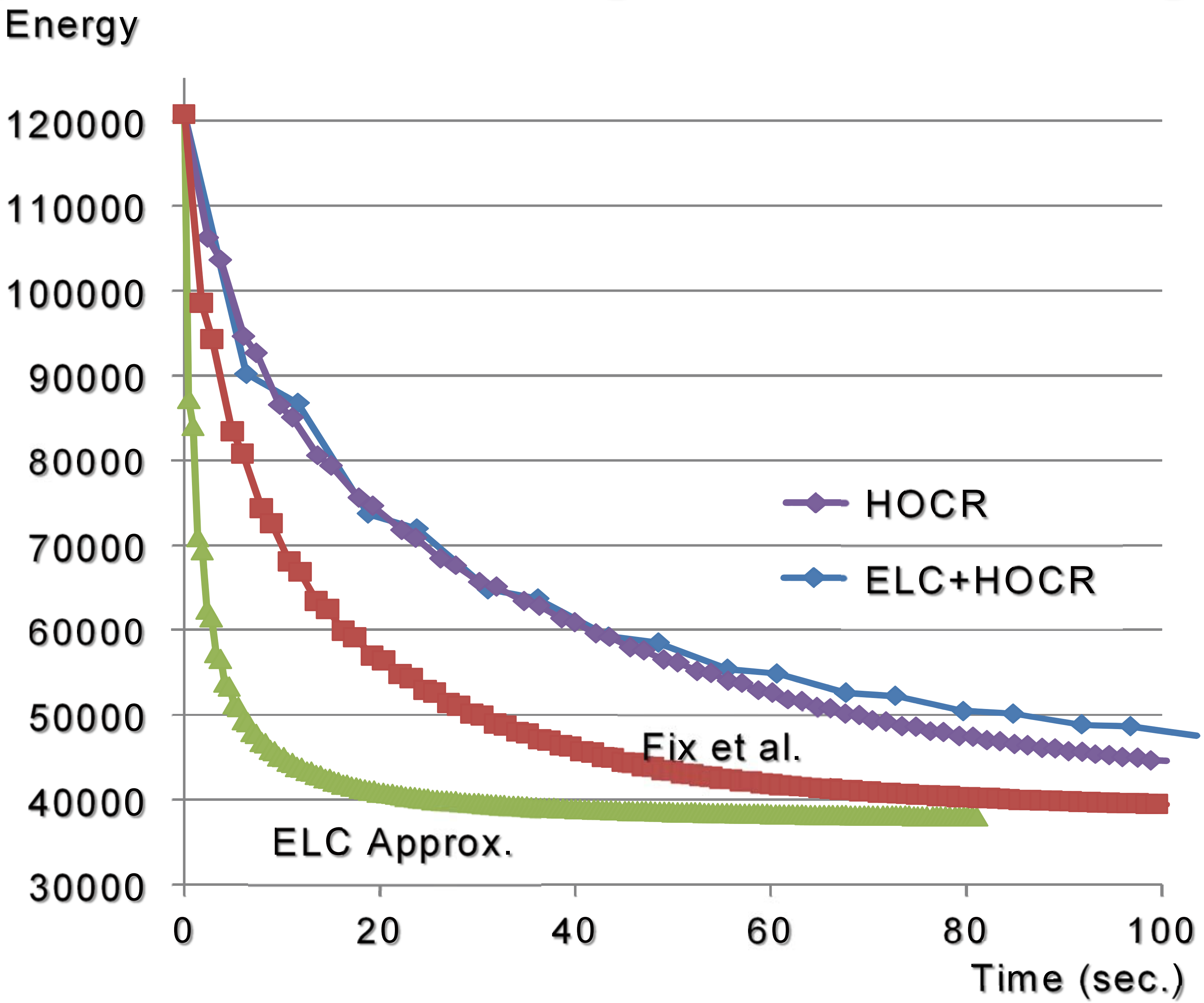
Experiment: 4th deg. FoE denoising



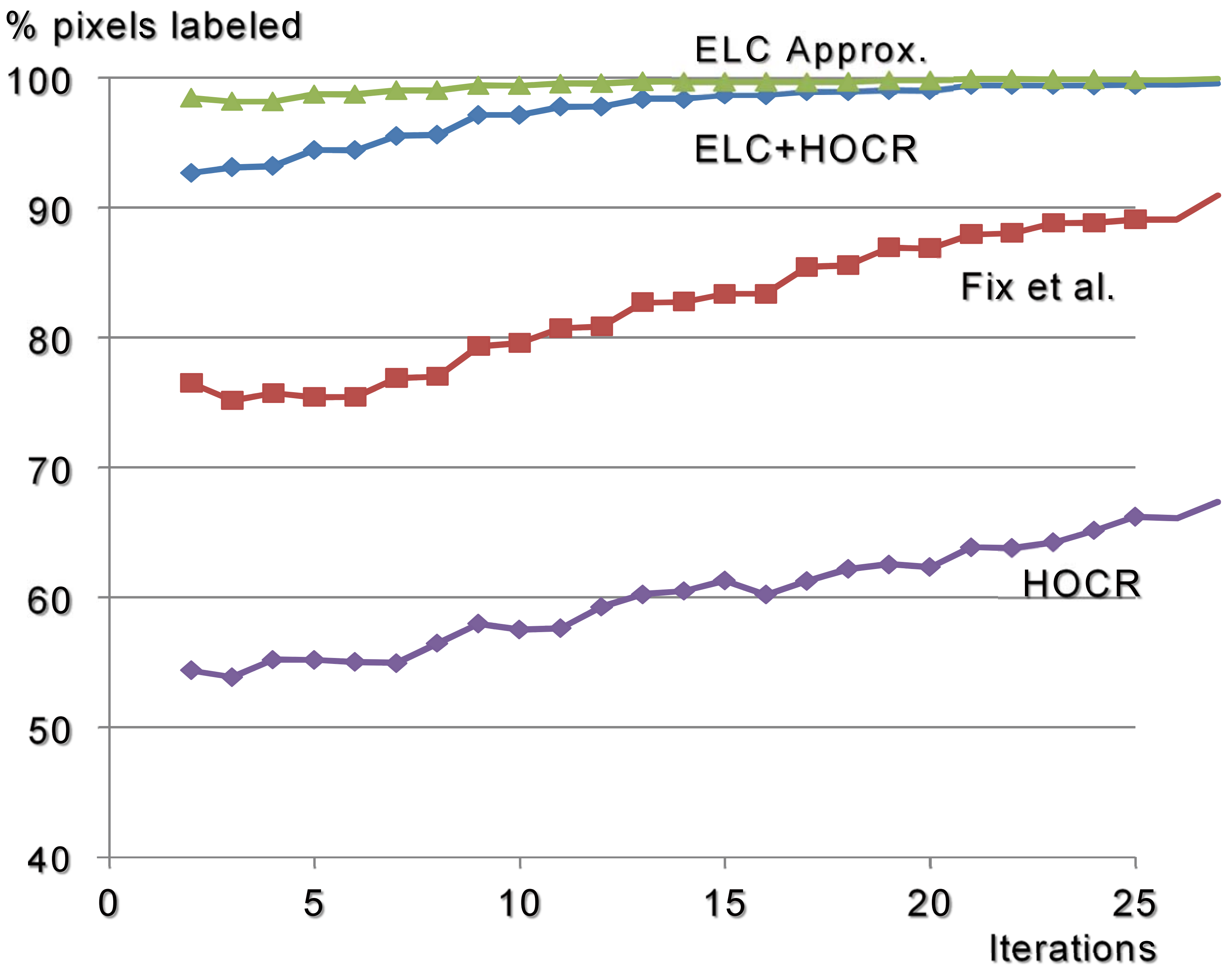
Experiment: 4th deg. FoE denoising



Experiment: 4th deg. FoE denoising



Experiment: 4th deg. FoE denoising



Experiment: 4th deg. FoE denoising

- ELC exists:
  - 3rd degree: 96.12%
  - 4th degree: 99.60%
- Approximation:
  - Guessed configuration is in fact an ELC
    - 3rd degree: 88% of the time
    - 4th degree: 97% of the time
  - Even if it is not an ELC, it is not part of maximizer
    - 3rd degree: 99.98%
    - 4th degree: 99.997%
  - 99.99988% of (labeled) pixels correctly labeled

Conclusion

- Higher-order energy minimization
  - Unary, Pairwise, Triple, ….
  - Binary labels: reduce to first order
    - Then use graph cuts
  - Multiple labels: use Fusion Move
- Reducing to first order
  - Before: add variables
  - New: no variables added
    - Faster
    - Much less memory
- Code available:
  http://www.f.waseda.jp/hfs/software.html